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Self-Similar Solutions of the Boltzmann Equation and Their Applications

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We consider a class of solutions of the Boltzmann equation with infinite energy. Using the Fourier-transformed Boltzmann equation, we prove the existence of a wide class of solutions of this kind. They fall into subclasses, labelled by a parameter a, and are shown to be asymptotic (in a very precise sense) to the self-similar one with the same value of a (and the same mass). Specializing to the case of a Maxwell-isotropic cross section, we give evidence to the effect that the only self-similar closed form solutions are the BKW mode and the two solutions recently found by the authors. All the self-similar solutions discussed in this paper are *eternal*, i.e., they exist for $-\infty < t < \infty$, which shows that a recent conjecture cannot be extended to solutions with infinite energy. Eternal solutions with finite moments of all orders, and different from a Maxwellian, are also studied. It is shown that these solutions cannot be positive. Moreover all such solutions (partly negative) must be asymptotically (for large negative times) close to the exact eternal solution of BKW type.

KEY WORDS: Boltzmann equation; eternal solutions, self-similar solutions.

1. INTRODUCTION

We consider a class of solutions of the Boltzmann equation which can be constructed starting from rather peculiar, self-similar solutions with infinite energy. We were led to considering these solution by the interesting question of extending the solution for the structure of an infinitely strong shock wave from the case of hard spheres (or cutoff potentials)[(1–3)] to that of molecules interacting at distance. This connection will be discussed in

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Section 2. We show that the problem can be formulated in terms of the space homogeneous BE. Moreover it is connected with the old question concerning large time asymptotics of solutions with infinite energy.

In Section 3 we give an accurate statement of the problem we are going to consider. In Section 4 we detail the Fourier-transformed Boltzmann equation and prove that its solution in the space homogeneous case exists and is unique in the class of locally bounded functions, for cutoff Maxwell molecules and $f_0 \in L^1_+$, without any further assumption. In Section 5 we describe a series representation for solutions with infinite second moments, and give estimates for the coefficients of the series, which guarantee its convergence. Among these solutions there are some which are self-similar. Apart from constants which are related to simple invariance properties, these solutions turn out to depend on a parameter α, taking values between 0 and 1. There is essentially one solution for each value of this parameter.

A similar representation was considered 25 years ago when one of the authors[4] constructed self-similar solutions with finite energy. These solutions obey the same equation as our solutions, but with $\alpha > 1$. Several authors (see refs. 5 and 6 for a review) considered these solutions in more detail and finally it was proved by Barnsley and Cornille[7] (for the simplest case) that all these solutions, except the so-called "BKW-mode,"[8–10] do not correspond to positive distribution functions. Thus those solutions do not appear to be useful for applications.

In Section 6 we state and prove the existence of the solutions discussed in the previous section and prove that a solution which is not self-similar is asymptotic (in a very precise sense) to the self-similar one with the same value of α. The main results of Sections 3–6 are given by Theorems 6.1–6.3.

In Section 7 we consider the case of a Maxwell-isotropic cross section and give evidence to the effect that the only self-similar closed form solutions are the BKW mode and the solution previously found by the authors.[11, 12] We also give an example of a positive initial datum for the functions represented by the series utilized in Sections 5 and 6.

The self-similar solutions discussed in this paper are *eternal*, i.e., they exist for $-\infty < t < \infty$, which shows that a recent conjecture[13] cannot be extended to solutions with infinite energy. We remark that multiplying any solution in the Fourier space by a Maxwellian, we obtain an eternal solution which tends to this Maxwellian at $t = -\infty$. This also makes the distribution function infinitely smooth (Section 8).

Section 8 is devoted to eternal solutions with finite moments of all orders. First we prove that such solutions cannot be positive. Then we study possible partly negative solutions and give ample evidence in favor of the fact, for large negative times, that they must be close to an exact self-similar solution of BKW type. This result may be considered as an unexpected answer to the old question related to the so-called Krook–Wu conjecture:[14] does this simple solution play any special (asymptotic) role? The answer we give is "yes," but in the non-physical domain of large negative times and partly negative solutions.

2. CONNECTION WITH THE SHOCK-WAVE PROBLEM.

Recently Grad's conjecture[1] that there is a solution of the Boltzmann equation describing the structure of an infinite Mach number shock wave, was successfully tested by numerical methods of both Monte Carlo[2] and deterministic nature.[3] For molecules without angular cutoff, the assumption by Grad (a delta plus a regular function) does not hold. Monte Carlo simulations[15] seem to indicate, however, that a solution exists in this case as well.

This revives the interest in a paper by Pomeau,[16] who proposed an approach where the delta term is replaced by an approximate self-similar solution. Pomeau's idea is crystal-clear but his exposition is not. Here we make his conjecture precise.

The idea is that there is an asymptotic solution for $x \to -\infty$ of the Boltzmann equation having the following form:

$$f(x, \mathbf{v}) = |x|^{3\nu} F(\mathbf{v}\,|x|^{\nu}\,\mathrm{sgn}(x))$$

where, if ξ is the molecular velocity and u_0 the bulk speed upstream, $\mathbf{v} = \xi - u_0 \mathbf{i}$, and F is normalized to a constant:

$$\int F(\mathbf{w})\, d\mathbf{w} = c$$

Then we also have

$$\int f(\mathbf{v})\, d\mathbf{v} = c$$

If $\nu > 0$, for $x \to -\infty$, f tends to a delta (multiplied by c). Let us insert this *ansatz* into the steady Boltzmann equation with plane symmetry:[17]

$$\xi_1 f_x = Q(f, f) \qquad (\xi_1 = \xi \cdot \mathbf{i})$$

Then we obtain for a power-law intermolecular potential $U(r) \cong r^{1-s}$:

$$(u_0 + v_1)(3\nu\,|x|^{3\nu-1}\,F + \nu\,|x|^{4\nu-1}\,\mathbf{v}\cdot F_{\mathbf{w}}) = Q(f, f)\,|x|^{3\nu(1-\frac{s-5}{s-1})}$$

where the differentiation in $F_{\mathbf{w}}$ and integration in the collision term are made with respect to the variable

$$\mathbf{w} = \mathbf{v}\,|x|^{\nu}\,\mathrm{sgn}(x)$$

Let us rewrite the equation in terms of w:

$$(u_0 + w_1\,|x|^{-\nu})(3\nu\,|x|^{3\nu-1}\,F + \nu\,|x|^{3\nu-1}\,\mathbf{w}) \cdot F_{\mathbf{w}}) = Q(F, F)\,|x|^{3\nu(1-\frac{s-5}{s-1})}$$

The fact that F is only an asymptotic and not an exact solution arises from the fact that we cannot make x disappear. In order to do that we must neglect the second term in the factor $(u_0 + w_1\,|x|^{-\nu})$. This term tends to zero for a fixed value of w when $x \to -\infty$. Then x disappears if we choose

$$\nu = \frac{s-1}{s-5}$$

In fact, we have:

$$u_0(3\nu F + \nu \mathbf{w} \cdot F_{\mathbf{w}}) = Q(F, F)$$

The method does not seem to work for Maxwell molecules ($s = 5$), but in this case one can make the alternative *ansatz*:

$$f(x, \mathbf{v}) = e^{3\lambda x} F(\mathbf{v} e^{\lambda x})$$

to obtain

$$(u_0 + w_1 e^{\lambda x})(3\lambda e^{3\lambda x} F + \lambda e^{3\lambda x} \mathbf{w} \cdot F_{\mathbf{w}}) = Q(F, F)\, e^{3\lambda x}$$

If we neglect again the second term in the first factor, we get rid of x:

$$u_0(3\lambda F + \lambda \mathbf{w} \cdot F_{\mathbf{w}}) = Q(F, F) \tag{2.1}$$

The approximation is not uniformly valid. This has the result that the second order moment cannot exist. In fact if we multiply by $|\mathbf{w}|^2$ and integrate formally, we have a contradiction.

We remark that, if we go back to the x and $\mathbf{v}$ variables, f now satisfies

$$u_0 f_x = Q(f, f)$$

which looks like the space homogeneous Boltzmann equation with time $t = x/u_0$, as already indicated by Pomeau,[16] who from this inferred that the solution should not have finite energy.

The last remark shows that the same equation as (2.1) follows from the homogeneous Boltzmann equation

$$f_t = Q(f, f)$$

when we let

$$f(t, \mathbf{v}) = e^{-3\lambda t} F(\mathbf{v} e^{-\lambda t})$$

Having thus motivated our interest in this kind of solutions, we start a detailed study of the solutions of the space homogeneous Boltzmann equation, with a specific concern for those which do not possess finite moments of second order.

3. MAXWELL MOLECULES. STATEMENT OF THE PROBLEM

Let $f(\mathbf{v}, t)$ (where $\mathbf{v} \in \Re^3$ and $t \in \Re_+$ are the velocity and time variables) be a distribution function normalized by

$$\int_{\Re^3} d\mathbf{v}\, f(\mathbf{v}) = 1 \tag{3.1}$$

and satisfying the homogeneous Boltzmann equation for Maxwell's molecules:

$$f_t = \int_{\Re^3 \times S^2} d\mathbf{w}\, d\mathbf{n}\, g\left(\frac{\mathbf{V} \cdot \mathbf{n}}{|\mathbf{V}|}\right) [f(\mathbf{v}')\, f(\mathbf{w}') - f(\mathbf{v})\, f(\mathbf{w})] \tag{3.2}$$

where, for simplicity, we do not indicate the time dependence of f in the collision term, and

$$\mathbf{V} = \mathbf{v} - \mathbf{w}, \qquad \mathbf{v}' = \tfrac{1}{2}(\mathbf{v} + \mathbf{w} + |\mathbf{V}|\, \mathbf{n}), \qquad \mathbf{w}' = \tfrac{1}{2}(\mathbf{v} + \mathbf{w} - |\mathbf{V}|\, \mathbf{n}), \qquad \mathbf{n} \in S^2$$

$g(\cos\theta)$ denotes the scattering cross section multiplied by $|\mathbf{V}|$ and, as such, it is independent of $|\mathbf{V}|$ for Maxwell molecules. The "true" cross section for Maxwell's potential (inversely proportional to r^{-4}) leads to the asymptotic behavior[17] $g(\mu) \cong (1-\mu)^{-5/4}$ as $\mu \to 1$. Therefore we sometimes consider a cutoff collision operator (pseudo-Maxwell molecules) with $g \in L^1(-1, 1)$ and normalize $g(\mu)$ is such a way that

$$\int_{S^2} d\mathbf{n}\, g\left(\frac{\mathbf{V} \cdot \mathbf{n}}{|\mathbf{V}|}\right) = 2\pi \int_{-1}^{1} g(\mu)\, d\mu = 1 \tag{3.3}$$

by an appropriate scaling of the time variable in (3.2). The conditions (3.1), (3.3) allow us to write Eq. (3.2) (in the cutoff case) and the corresponding initial condition as

$$f_t = Q_+(f, f) - f, \qquad f_{|t=0} = f_0 \tag{3.4}$$

where $f_0 \in L^1_+$ is normalized by Eq. (3.1).

The solution of the initial value problem (3.4) is given by the so-called Wild sum:[18]

$$f(\mathbf{v}, t) = e^{-t} \sum_{k=0}^{\infty} (1 - e^{-t})^n f_n(\mathbf{v}) \tag{3.5}$$

where f_0 is the initial distribution function in (3.4) and

$$f_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} Q_+(f_k, f_{n-k}), \qquad n = 0, 1,... \tag{3.6}$$

Then it can be easily shown that the series (3.5) converges for any $t > 0$ and any $f_0 \in L^1_+(\mathfrak{R}^3)$; moreover $f \in L^1_+(\mathfrak{R}^3)$ and satisfies the condition in (3.1). The solution for true Maxwell molecules can be constructed as the limit of the previous solution for a vanishing angular cutoff. It is clear that a solution of this kind can also be defined for any initial condition $f_0 \in L^1_+(\mathfrak{R}^3)$. The spatially homogeneous problem for Maxwell molecules has been previously studied in detail (see, in particular, the review of ref. 19). However, to the best of our knowledge, one interesting question was never addressed before. The usual restriction of the type

$$\int_{\mathfrak{R}^3} d\mathbf{v}\, f_0(\mathbf{v})(1 + |\mathbf{v}|^2) = 1 \tag{3.7}$$

is obviously not necessary for the solution (3.5)–(3.6) to exist: $f_0 \in L^1_+(\mathfrak{R}^3)$ is enough. On the other hand, it is well-known that the restriction (3.7) guarantees that the solution $f(\mathbf{v}, t)$ tends, as $t \to \infty$, to a Maxwellian distribution. Let us assume now that $f_0 \in L^1_+(\mathfrak{R}^3)$, but

$$\int_{\mathfrak{R}^3} d\mathbf{v}\, f_0(\mathbf{v})\, |\mathbf{v}|^2 = \infty \tag{3.8}$$

What happens with the solution $f(\mathbf{v}, t)$ as $t \to \infty$ in this case? This question will be considered below in some particular cases. The self-similar solutions arise quite naturally in this case as asymptotic states for $t \to \infty$.

4. FOURIER TRANSFORM AND UNIQUENESS LEMMA

Let

$$\hat{f}(\mathbf{k}, t) = \int_{\Re^3} d\mathbf{v}\, f(\mathbf{v}, t)\, e^{-i\mathbf{k}\cdot\mathbf{v}}, \qquad \mathbf{k} \in \Re^3 \tag{4.1}$$

then the normalization in Eq. (3.1) becomes

$$\hat{f}_0 = 1 \tag{4.2}$$

and the corresponding initial value problem reads[6]

$$\hat{f}_t = \int_{S^2} d\mathbf{n}\, g\left(\frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|}\right)[\hat{f}(\mathbf{k}_+)\,\hat{f}(\mathbf{k}_-) - \hat{f}(\mathbf{0})\,\hat{f}(\mathbf{k})], \qquad \hat{f}_{|t=0} = \hat{f}_0(\mathbf{k}) \tag{4.3}$$

where

$$\mathbf{k}_\pm = \tfrac{1}{2}(\mathbf{k} \pm |\mathbf{k}|\,\mathbf{n}), \qquad \mathbf{n} \in S^2$$

In the cutoff case (3.3) we similarly obtain

$$\hat{f}_t = \hat{Q}_+(\hat{f}, \hat{f}) - \hat{f}, \qquad \hat{f}_{|t=0} = \hat{f}_0 \tag{4.4}$$

Then one can construct the solution by a series similar to Wild's sum (3.5)–(3.6). In this way we show that the solution $\hat{f}(\mathbf{k}, t)$ of the problem (4.4) exists for any initial characteristic function (Fourier transform of a probability measure) $\hat{f}_0(\mathbf{k})$ and that $\hat{f}(\mathbf{k}, t)$ is also a characteristic function for any $t > 0$.

There are other methods, however, to construct the solution of the problem defined in (4.4). In order to be sure that all the methods lead to the same solution, we shall later need the following uniqueness result.

Lemma 4.1. If $\hat{f}_0(\mathbf{k})$ is a characteristic function, then the solution $\hat{f}(\mathbf{k}, t)$ of the problem (4.4) is unique in the class of functions satisfying the inequality

$$\|\hat{f}\|_{R,T} = \sup_{|\mathbf{k}| \leqslant R,\, 0 \leqslant t \leqslant T} |\hat{f}(\mathbf{k}, t)| < \infty \tag{4.5}$$

for any $R > 0$ and $T > 0$.

Proof. We first note that $|\hat{f}_0(\mathbf{k})| \leqslant \hat{f}_0(0) = 1$ and construct a solution $\hat{f}_W(\mathbf{k}, t)$ given by Wild's sum

$$\hat{f}_W(\mathbf{k}, t) = e^{-t} \sum_{k=0}^{\infty} (1-e^{-t})^n \hat{f}_n(\mathbf{v}) \tag{4.6}$$

where $\hat{f}_0$ is the initial characteristic function in (4.4) and

$$\hat{f}_{n+1} = \frac{1}{n+1} \sum_{j=0}^{n} \hat{Q}_+(\hat{f}_j, \hat{f}_{n-j}), \qquad n = 0, 1,... \tag{4.7}$$

$$\hat{Q}_+(\hat{f}, \hat{g}) = \int_{S^2} d\mathbf{n}\, g\left(\frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|}\right) \hat{f}(\mathbf{k}_+)\, \hat{g}(\mathbf{k}_-) \tag{4.8}$$

Then $\hat{f}_W(\mathbf{k}, t)$ is a characteristic function for any $t > 0$ and hence $|\hat{f}_W(\mathbf{k}, t)| \leqslant 1$. Let us assume now that there exists another solution $\hat{f}(\mathbf{k}, t)$ of the problem (4.4), satisfying the condition (4.5). Then the function $h = \hat{f}(\mathbf{k}, t) - \hat{f}_W(\mathbf{k}, t)$ is a solution of the following problem

$$h_t + h = \hat{P}h, \qquad h_{|t=0} = 0 \tag{4.9}$$

where the linear operator $\hat{P} = \hat{P}(t)$ is given by

$$\hat{P}h = \int_{S^2} d\mathbf{n}\, g_*\left(\frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|}\right)[\hat{f}_W(\mathbf{k}_+, t) + \hat{f}(\mathbf{k}_+, t)]\, h(\mathbf{k}_-, t) \tag{4.10}$$

where

$$g_*(\mu) = \tfrac{1}{2}[g(\mu) + g(-\mu)]$$

Eqs. (4.9) lead to

$$h(\mathbf{k}, t) = \int_0^t d\tau\, e^{-(t-\tau)}[\hat{P}h](\mathbf{k}, t) \tag{4.11}$$

We denote for an arbitrary $R > 0$

$$\|h\|_R\,(t) = \sup_{|\mathbf{k}| \leqslant R} |h|\,(\mathbf{k}, t)$$

then, since

$$|\hat{f}_W(\mathbf{k}, t)| \leqslant 1, \qquad \int_{S^2} d\mathbf{n}\, g_*\left(\frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|}\right) = 1$$

we have, for any $0 \leqslant t \leqslant T$,

$$\sup_{|\mathbf{k}| \leqslant R} [\hat{P}h](\mathbf{k}, t) \leqslant [1 + \|\hat{f}\|_{R,T}] \, \|h\|_R \, (t)$$

Therefore Eq. (4.10) leads to

$$\|h\|_R \, (t) \leqslant [1 + \|\hat{f}\|_{R,T}] \int_0^t d\tau \, \|h\|_R \, (\tau), \qquad 0 \leqslant t \leqslant T$$

It is well known (Gronwall's lemma) that this inequality has the only trivial solution $\|h\|_R \, (t) = 0$. Hence $h(\mathbf{k}, t) = 0$ for any $|\mathbf{k}| \leqslant R$ and any $0 \leqslant t \leqslant T$. However R and T are arbitrary positive numbers and therefore $h = \hat{f}(\mathbf{k}, t) - \hat{f}_W(\mathbf{k}, t) = 0$ in $\Re^3 \times \Re_+$. This completes the proof.

5. SPECIAL CLASSES OF SOLUTIONS

The aim of this section is to construct some solutions with infinite second moments in a more explicit form and to study them in more detail.

We consider isotropic distribution functions $f(|\mathbf{v}|, t)$ and the corresponding characteristic functions

$$\hat{f}(|\mathbf{k}|, t) = \phi(x, t), \qquad x = |\mathbf{k}|^2/2 \tag{5.1}$$

Then the equation for $\phi(x, t)$ reads

$$\phi_t = \int_0^1 ds \, G(s)[\phi(sx) \, \phi((1-s) \, x) - \phi(0) \, \phi(x)] \tag{5.2}$$

where

$$G(s) = 4\pi g(1 - 2s), \qquad 0 \leqslant s \leqslant 1 \tag{5.3}$$

Noting that

$$\phi(0, t) = \int_{\Re^3} d\mathbf{v} \, f(\mathbf{v}) = 1; \qquad \phi'(0, t) = -\tfrac{1}{3} \int_{\Re^3} d\mathbf{v} \, f(\mathbf{v}) \, |\mathbf{v}|^2 \tag{5.4}$$

we see that an infinite second moment (3.8) corresponds to the case $\phi'(x) \to -\infty$ as $x \to 0+$. The typical behavior of characteristic functions of this kind near the origin is described by the following asymptotic formula:

$$\phi(x) = 1 - ax^{\alpha}[1 + O(x^{\alpha})], \qquad x \to 0^+, \quad 0 < \alpha < 1, \quad a > 0 \tag{5.5}$$

Having in mind the usual class of rapidly decreasing functions[19] with $\alpha = 1$, we extend this class to real positive values of α by letting:

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}, \qquad \alpha > 0 \tag{5.6}$$

Such solutions were considered earlier[4] for $\alpha > 1$; here we shall use a similar approach for $0 < \alpha < 1$. A peculiar aspect of this case is that we cannot expect relaxation to any steady state as $t \to \infty$. If one looks for the solution in the form (5.6) and substitute the series (5.6) into Eq. (5.2), then the first coefficients can be found immediately:

$$\phi_0(t) = 1, \qquad \phi_1(t) = \phi_1(0)\, e^{\lambda_\alpha t} \tag{5.7}$$

where

$$\lambda_\alpha = \lambda(\alpha) = \int_0^1 ds\, G(s)[s^\alpha + (1-s)^\alpha - 1] > 0, \qquad 0 < \alpha < 1 \tag{5.8}$$

In other words, the solution (5.6) with $0 < \alpha < 1$ behaves for small x like

$$\phi(x, t) \cong 1 - a x^\alpha e^{\lambda_\alpha t} = 1 - a(x e^{\mu_\alpha t})^\alpha, \qquad \mu_\alpha = \frac{\lambda_\alpha}{\alpha}$$

Therefore it is convenient to fix a certain value $0 < \alpha < 1$ and to represent the corresponding solution (5.6) in the form:

$$\phi(x, t) = \psi(x e^{\mu_\alpha t}, t), \qquad \mu_\alpha = \frac{\lambda_\alpha}{\alpha}, \qquad 0 < \alpha < 1 \tag{5.9}$$

where λ_α is given by (5.8). The function $\psi(x, t)$ obviously satisfies the equation

$$\psi_t + \mu_\alpha x \psi_x = \int_0^1 ds\, G(s)[\psi(sx)\, \psi((1-s)\, x) - \psi(0)\, \psi(x)] \tag{5.10}$$

with the same initial condition

$$\psi_{|t=0} = \phi_{|t=0} = \phi_0 \tag{5.11}$$

Substituting the series

$$\psi(x, t) = \sum_{n=0}^{\infty} \psi_n(t) \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} \tag{5.12}$$

into the equation (5.10), we obtain the following set of recurrence equations:

$$\begin{gathered}\frac{d\psi_0}{dt} = \frac{d\psi_1}{dt} = 0 \\ \frac{d\psi_n}{dt} + \gamma_n(\alpha)\,\psi_n = \sum_{j=1}^{n-1} B_\alpha(j, n-j)\,\psi_j \psi_{n-j}, \qquad n = 2, 3,\ldots\end{gathered} \tag{5.13}$$

where

$$\begin{aligned}\gamma_n(\alpha) &= n\alpha\mu_\alpha - \lambda(n\alpha) = n\lambda_\alpha - \lambda(n\alpha) \\ \lambda(p) &= \int_0^1 ds\, G(s)[s^p + (1-s)^p - 1] \\ B_\alpha(j, l) &= \frac{\Gamma(n\alpha+1)}{\Gamma(j\alpha+1)\,\Gamma(l\alpha+1)} \int_0^1 ds\, G(s)\, s^{j\alpha}(1-s)^{l\alpha}, \qquad n \geqslant 2\end{aligned} \tag{5.14}$$

Noting that $\lambda'(p) < 0$ for any $p > 0$, we get the following estimates:

$$\gamma_n(\alpha) = n\lambda(\alpha) - \lambda(n\alpha) \geqslant (n-1)\,\lambda(\alpha) \tag{5.15}$$

and $\gamma_n(\alpha) > 0$ if $n \geqslant 2$. The recurrence relations for the coefficients $\psi_n(t)$ follow from Eq. (5.13):

$$\psi_n(t) = \psi_n(0)\, e^{-\gamma_n(\alpha)\, t} + \sum_{j=1}^{n-1} B_\alpha(j, n-j) \int_0^t d\tau\, e^{-\gamma_n(\alpha)(t-\tau)} \psi_j(\tau)\, \psi_{n-j}(\tau),$$
$$n = 2, 3,\ldots \tag{5.16}$$

It is obvious from these formulas and from Eqs. (5.14) that

$$\psi_n(t) \to_{t\to\infty} u_n, \qquad n = 0, 1,\ldots \tag{5.17}$$

where $\{u_n\}$ is the unique steady solution of (5.13) given by the recurrence formulas:

$$u_0 = 1,\ u_1 = \psi_1,\ u_n = \frac{1}{\gamma_n(\alpha)} \sum_{j=1}^{n-1} B_\alpha(j, n-j)\, u_j u_{n-j}, \qquad n = 2, 3,\ldots \tag{5.18}$$

Until now our considerations on Eq. (5.4) have been quite formal since nothing was said about the convergence of the series (5.12). In order to make it rigorous we assume that there exists a number A such that

$$|\psi_n(0)| \leqslant A^n, \qquad n = 1, 2,... \tag{5.19}$$

and try to prove a similar estimate (with another number $A_1 \geqslant A$) for $\psi_n(t)$ uniformly for $t \in [0, \infty)$. An estimate of this kind guarantees the convergence of the series (5.12) for all $x, t \geqslant 0$.

To this end, we first prove the following lemma, providing a useful estimate, holding, in particular, for true Maxwellian molecules.

Lemma 5.1. If $0 \leqslant G(s) \leqslant as^{-(1+\gamma)}$ for some $a > 0$ and $0 < \gamma < 1$, then there exists a number $R = R(\alpha, \gamma)$, such that, for any $\alpha > \gamma$,

$$\frac{1}{n-1} \sum_{j=1}^{n-1} B_\alpha(j, n-j) \leqslant aR(\alpha, \gamma), \qquad n = 2, 3,... \tag{5.20}$$

where the coefficients $B_\alpha(j, l)$ are given by Eqs. (5.14).

Proof. We shall use below two well-known formulas for the Beta- and Gamma-functions:

$$\int_0^1 ds\, s^{x-1}(1-s)^{y-1} = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \qquad \lim_{z\to\infty} \frac{\Gamma(z)\, z^a}{\Gamma(z+a)} = 1 \tag{5.21}$$

The first of these formulas combined with the assumptions of the lemma leads to the following inequality:

$$B_\alpha(j, n-j) \leqslant a \frac{\Gamma(j\alpha-\gamma)\,\Gamma(n\alpha+1)}{\Gamma(j\alpha+1)\,\Gamma(n\alpha+1-\gamma)}$$

hence

$$\frac{1}{n-1} \sum_{j=1}^{n-1} B_\alpha(j, n-j) \leqslant a \frac{\Gamma(n\alpha+1)}{(n-1)\,\Gamma(n\alpha+1-\gamma)} \sum_{j=1}^{n-1} \frac{\Gamma(j\alpha-\gamma)}{\Gamma(j\alpha+1)} \qquad n = 2, 3,... \tag{5.22}$$

The second formula in (5.21) shows that

$$\frac{\Gamma(j\alpha-\gamma)}{\Gamma(j\alpha+1)} \cong_{j\to\infty} (j\alpha)^{-(1+\gamma)}, \qquad \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\gamma)} \cong_{n\to\infty} (n\alpha)^{\gamma}$$

Therefore

$$S(\alpha, \gamma) = \sum_{j=1}^{\infty} \frac{\Gamma(j\alpha - \gamma)}{\Gamma(j\alpha + 1)} < \infty$$

$$q(\alpha, \gamma) = \sup_{n=2, 3,...} \frac{\Gamma(n\alpha + 1)}{(n-1)\,\Gamma(n\alpha + 1 - \gamma)} < \infty$$

Substituting these estimates into inequality (5.22), we obtain inequality (5.20) with $R = qS$ and the lemma is proved.

Remark. The case of true Maxwell molecules corresponds to $\gamma = 1/4$.

Now we can easily obtain the estimates we need for $\psi_n(t)$, as given by Eqs. (5.16).

Lemma 5.2. If $0 \leqslant G(s) \leqslant as^{-(1+\gamma)}$ for some $a > 0$ and $0 < \gamma < 1$, and the inequalities (5.19) hold for some $A > 0$, then for any $\alpha > \gamma$ and any $t \geqslant 0$

$$|\psi_n(t)| \leqslant A^n \left[1 + \frac{a}{\lambda(\alpha)} R(\alpha, \gamma) \right]^{n-1}, \qquad n = 1, 2,... \tag{5.23}$$

where $\lambda(\alpha)$ is given by Eq. (5.14), $R(\alpha, \gamma)$ is defined in the previous lemma.

Proof. The estimate (5.23) is obvious for $n = 1$ since $\psi_1(t) = \psi_1(0)$ according to Eq. (5.16). Therefore we proceed by induction and assume the estimate (5.23) for $n = 1, 2,..., m-1$ with an arbitrary $m \geqslant 2$. Then Eq. (5.16) with $n = m$ yields

$$|\psi_m(t)| \leqslant A^m \left[e^{-\gamma_m(\alpha)\, t} + b^{m-2} \sum_{j=1}^{m-1} B_\alpha(j, m-j)\, \frac{1 - e^{-\gamma_m(\alpha)\, t}}{\gamma_m(\alpha)} \right]$$

where

$$b = 1 + \frac{a}{\lambda(\alpha)} R(\alpha, \gamma), \qquad 1 > \alpha > \gamma > 0$$

The inequality (5.15) for $\gamma_m(\alpha)$ and obvious estimates for $e^{-\gamma_m(\alpha)\,t}$ lead to

$$|\psi_m(t)| \leqslant A^m \left[1 + \frac{b^{m-2}}{\lambda(\alpha)} \frac{1}{m-1} \sum_{j=1}^{m-1} B_\alpha(j, m-j) \right]$$
$$\leqslant A^m \left[1 + \frac{b^{m-2}}{\lambda(\alpha)} aR(\alpha, \gamma) \right]$$

where the second inequality follows from the previous lemma. Hence

$$|\psi_m(t)| \leqslant A^m[1 + b^{m-2}(b-1)] \leqslant A^m b^{m-1}, \qquad m \geqslant 2$$

since $b > 1$. This completes the proof.

A similar result can be obtained for the coefficients $\{u_n\}$ given by the recurrence relations (5.18).

Lemma 5.3. Under the assumptions of Lemma 5.1 the following estimate is valid

$$|u_n(t)| \leqslant |u_1|^n \left[\frac{a}{\lambda(\alpha)} R(\alpha, \gamma) \right]^{n-1}, \qquad n = 1, 2,... \tag{5.24}$$

The proof is merely a simplified repetition of the proof of Lemma 5.2.

Lemma 5.2 gives sufficient restrictions on $G(s)$ and on the initial data $\phi(x, 0) = \phi_0(x)$ for the solution $\phi(x, t)$ of Eq. (5.2) to be represented by a series of the form (5.6) convergent for all $x > 0$ uniformly on $t \in [0, \infty)$. Some asymptotic properties of the solution are studied in Section 6.

6. SELF-SIMILAR SOLUTIONS AS ASYMPTOTIC STATES

The main results of Section 5 can be formulated in the following way. We consider Eq. (5.2) and assume that

(A) There exist two numbers $a > 0$ and $0 < \gamma < 1$ such that

$$0 \leqslant G(s) \leqslant as^{-(1+\gamma)}, \qquad 0 < s < 1 \tag{6.1}$$

(B) the initial condition has the following form:

$$\phi(x, 0) = \phi_0(x) = \sum_{n=0}^{\infty} \phi_n^{(0)} \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)}, \qquad \phi_0^{(0)} = 1, \quad \phi_1^{(0)} \neq 0 \tag{6.2}$$

with some $\gamma < \alpha < 1$ and such that

$$\sup_{n=1,2,\ldots} |\phi_n^{(0)}|^{1/n} < \infty$$

Theorem 6.1. Under assumptions (A) and (B) there exists a unique solution of Eq. (5.2) satisfying the initial condition (6.2) and represented in the form

$$\phi(x, t) = \psi(xe^{\mu_\alpha t}, t), \qquad \mu_\alpha = \frac{1}{\alpha} \int_0^1 ds\, G(s)[s^\alpha + (1-s)^\alpha - 1] > 0 \tag{6.3}$$

where $\psi(x, t)$ is given by formulas (5.12), (5.16). The series (5.12) converges for all $x > 0$; moreover, $\sup_{n=1,2,\ldots} |\psi_n(t)|^{1/n} < a$ for all $t > 0$ and some constant a depending only on $G(s)$ and on the initial conditions.

The proof follows from the above described construction of the solution and from Lemma 5.2. The solution is unique, by the standard uniqueness theorem on ODEs and is given, by construction, by (5.12), (5.16).

Our second result concerns self-similar solutions and their connection with the solutions described by Theorem 6.1.

Theorem 6.2. If $G(s)$ satisfies assumption (A) with some $\gamma > 1$, then for any number μ_α given by (6.3) with $\gamma < \alpha < 1$, there exists a self-similar solution $\phi(x, t) = u^{(\alpha)}(xe^{\mu_\alpha t})$ given by the following series:

$$u^{(\alpha)}(x) = \sum_{n=0}^{\infty} u_n^{(\alpha)} \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)}, \qquad \sup_{n=1,2,\ldots} |u_n^{(\alpha)}|^{1/n} < \infty \tag{6.4}$$

where $u_0^{(\alpha)} = 1$, $u_1^{(\alpha)} \neq 0$ can be chosen arbitrarily and $u_n^{(\alpha)}$ $(n \geqslant 2)$ are given by the recurrence formulas in (5.18). If $\psi(x, t)$ is the function defined in Theorem 6.1 (for a given value of α), then

$$\lim_{t \to \infty} \psi(x, t) = u^{(\alpha)}(x) \tag{6.5}$$

for any $x \geqslant 0$, provided $u_1^{(\alpha)} = \psi_1(0)$.

The proof is based on the construction described in the previous section (the function $u^{(\alpha)}$ is obviously a steady solution of Eq. (5.10)) and on Lemmas 5.2 and 5.3. It was already shown in Section 5 (see Eq. (5.17)) that $u^{(\alpha)}(x)$ is formally the limit of the series (5.12) as $t \to \infty$. On the other hand, for any $x > 0$ the series converges uniformly on $t \in [0, \infty)$ (Lemma 5.2).

Therefore the limit is rigorously justified. The inequality (6.4) follows from Lemma 5.3. Thus the theorem is proved.

Theorem 6.2 explains the exact meaning of the asymptotic equality

$$\phi(x, t) = \psi(xe^{\mu_\alpha t}, t) \cong_{t\to\infty} u^{(\alpha)}(xe^{\mu_\alpha t}) \tag{6.6}$$

and the corresponding relation

$$f(|\mathbf{v}|, t) = e^{-\frac{3}{2}\mu_\alpha t} F(|\mathbf{v}|, t)\, e^{-\frac{1}{2}\mu_\alpha t}, t) \cong_{t\to\infty} e^{-\frac{3}{2}\mu_\alpha t} \Phi_\alpha(|\mathbf{v}|\, e^{-\frac{1}{2}\mu_\alpha t}) \tag{6.7}$$

where

$$\int_{\mathfrak{R}^3} d\mathbf{v}\, \Phi_\alpha(|\mathbf{v}|, t)\, e^{-i\mathbf{k}\cdot\mathbf{v}} = u^{(\alpha)}(|\mathbf{k}|^2/2) \tag{6.8}$$

for the corresponding solutions of the Boltzmann equation (3.2). We stress that the transition from (6.6) to Eq. (6.7) is not justified as yet, since we still must prove that $u^{(\alpha)}(|\mathbf{k}|^2/2)$ is a characteristic function.

In order to do this, we first assume $G(s)$ to be bounded and use the uniqueness result proved in Section 5. Lemma 5.1 shows the following: if any characteristic function $\phi_0(|\mathbf{k}|^2/2)$ can be represented in the form (6.2) with $x = |\mathbf{k}|^2/2$, then the corresponding solution $\phi(x, t)$ is a characteristic function at any $t > 0$, and so is $\psi(x, t)$ given by (6.3). On the other hand $u^{(\alpha)}(x)$ is a pointwise limit of $\psi(x, t)$ as $t \to \infty$, and is obviously continuous at $x = 0$. These properties are sufficient to establish that $u^{(\alpha)}(x)$ is also a characteristic function.[20]

The case of true Maxwell molecules (Lemma 4.1 was proved only for integrable cross sections) can be considered in a similar way. First we approximate $G(s)$ by integrable functions, say

$$G_\epsilon(s) = G(s) \quad \text{if } s > \epsilon, \qquad G_\epsilon(s) = 0 \quad \text{otherwise}$$

with $\epsilon \to 0$, and construct (for a given $\alpha > \gamma = 1/4$) the corresponding characteristic functions $u^{(\alpha)}(x; \epsilon)$ given by the series (6.4). Then each coefficient of the series is obviously a continuous function of ϵ at $\epsilon = 0$. Therefore we obtain a formal limit: $u_\epsilon \to_{\epsilon\to 0} u(x)$. To make it rigorous it is enough to find a uniform estimate (in ϵ) of the ratio $R_\epsilon(\alpha, \gamma)/\lambda_\epsilon(\alpha)$ in the equality (5.24). This can be easily done since $R_\epsilon(\alpha, \gamma)$ and $\lambda_\epsilon(\alpha)$ are monotonously increasing functions of ϵ and are continuous at $\epsilon = 0$. Hence the convergence is uniform in ϵ and taking the limit is justified. The final step is again to notice that $u^{(\alpha)}(x)$ is continuous at $x = 0$ and therefore $u^{(\alpha)}(x)$ is a characteristic function. The same consideration holds for any $G(s)$ satisfying the condition (6.1) with some $0 < \gamma < 1$.

To conclude we must prove that for any $0 < \alpha < 1$ there exists at least one characteristic function (initial condition for (5.2)) represented in the form (6.2). This can be done by a single example constructed in Section 7.

Finally we are able to formulate the result

Theorem 6.3. All self-similar solutions described in Theorem 6.2 are characteristic functions (Fourier transform of probability measures).

The proof was already given above except for the concrete example (Mittag–Leffler's function), which will be presented at the end of Section 7.

Remark. It seems quite probable that the corresponding self-similar solutions (see (6.8))

$$f(|\mathbf{v}|, t) = e^{-\frac{3}{2}\mu_\alpha t}\Phi_\alpha(|\mathbf{v}|\, e^{-\frac{1}{2}\mu_\alpha t})$$

of the Boltzmann equation are L^1 functions, not measures. This conjecture will be justified for some particular cases in Section 7. Our arguments are not sufficient, however, to prove the conjecture in the general case.

We can obviously extend all the results of this section to a class of initial conditions

$$\phi_0(x, \theta) = \phi_0(x)\, e^{-\theta x}$$

where $\phi_0(x)$ is given by Eq. (6.2). Then the solution is given by the equality

$$\phi(x, t; \theta) = e^{-\theta x}\phi(x, t; 0)$$

and has the asymptotics

$$\phi(x, t; \theta) = e^{-\theta x}\psi(x, t; \theta) \cong_{t\to\infty} e^{-\theta x}u^{(\alpha)}(xe^{\mu_\alpha t})$$

which follows directly from (6.6). The functions

$$w_{\alpha,\theta}(x, t) = e^{-\theta x}u^{(\alpha)}(xe^{\mu_\alpha t}), \qquad x = |\mathbf{k}|^2/2$$

represent a two-parameter ($\theta > 0$, $\gamma < \alpha < 1$) family of solutions invariant with respect to an obvious semigroup. The corresponding solutions of the Boltzmann equations exist and are non-negative as follows from Theorem 6.3. All these invariant solutions are eternal, i.e., they satisfy the Boltzmann equation for all $t \in (-\infty, \infty)$.

7. EXACTLY SOLVABLE CASES

Following our previous papers,[11, 12] we consider in this section the special case $G(s)=1$ in Eq. (5.2). Our aim here is to study all possible closed-form self-similar solutions in this case. As a by-product, we also present an example of a characteristic function having the form (6.2) with an arbitrary $0<\alpha<1$. This example completes the proof of Theorem 6.3.

If $G(s)=1$, Eq. (5.2) for self-similar solutions $\phi(x,t)=u(xe^{\mu t})$ leads to

$$\mu u'(x)=\frac{1}{x}\int_0^x dy\,u(y)\,u(x-y)-u(x), \qquad u(0)=1 \tag{7.1}$$

Let

$$y(p)=p\mathscr{L}[u]=p\int_0^\infty dx\,u(x)\,e^{-px} \tag{7.2}$$

then we obtain a nonlinear ODE:[11, 12]

$$\mu p^2 y''-py'+y(1-y)=0, \qquad y(p)\to_{p\to\infty} 1 \tag{7.3}$$

The case $\mu=0$ is trivial; thus we assume $\mu\neq 0$. Then Eq. (7.3) can be reduced to a standard form by the following substitutions

$$\begin{gathered}\text{(a)}\quad \mu\neq -1: \qquad y(p)=\frac{1}{2}-\frac{1}{A}\left[\frac{1}{2}-p^{2\beta}w(p^\beta)\right],\\ \beta=\frac{1+\mu}{5\mu}, \qquad A=\frac{1}{6\mu\beta^2}=\frac{25\mu}{6(1+\mu)^2};\end{gathered} \tag{7.4}$$

$$\text{(b)}\quad \mu=-1: \qquad y(p)=\frac{1}{2}-6w(\log p) \tag{7.5}$$

These substitutions lead to two equations for $w(t)$ (we omit the intermediate calculations)

$$\text{(a)}\quad \mu\neq -1: \qquad w''=6w^2+3\,\frac{1-A^2}{2t^4} \tag{7.6}$$

$$\text{(b)}\quad \mu=-1: \qquad w''=6w^2-\frac{1}{24} \tag{7.7}$$

Each of these equations has two particular solutions easy to find:

$$\text{(a)}\quad w=\frac{1\pm A}{2t^2};\qquad \text{(b)}\quad w=\pm\frac{1}{12}$$

These solutions correspond to the trivial ones ($y=0$, $y=1$) of the original equation.

The case $\mu=-1$ and the special case $A=\pm 1$ for $\mu\neq 1$ lead to the following first order ODEs:

$$\text{(a)}\quad (w')^2=4w^3+\text{const.};\qquad \text{(b)}\quad (w')^2=4w^3-\frac{w}{12}+\text{const.} \tag{7.8}$$

In turn, these equations can be solved in terms of the Weierstrass elliptic function $\mathscr{P}(t;g_2,g_3)$ satisfying the equation[21] (p. 629):

$$(\mathscr{P}')^2=4\mathscr{P}^3-g_2\mathscr{P}-g_3 \tag{7.9}$$

If $|A|\neq 1$ in Eq. (7.6), then the equation is not of the Painlevé type (see ref. 22). It has moving logarithmic critical points and therefore does not have any "simple" analytic solution.

Hence relatively simple, closed form solutions exist only for values of μ satisfying the equality

$$A^2=\frac{625\mu^2}{36(1+\mu)^4}=1 \tag{7.10}$$

and for $\mu=-1$. Thus there are five cases solvable in a closed form:

$$\mu=-1;\qquad \mu=2/3;\qquad \mu=3/2;\qquad \mu=-6;\qquad \mu=-1/6 \tag{7.11}$$

Not all these values of μ correspond to true solutions of the Boltzmann equation (7.1) since we also need the correct asymptotics (7.3) for $p\to\infty$. The qualitative behavior of the solutions of Eq. (7.6) with $|A|=1$ and (7.7) is best understood by an analogy with classical mechanics. We re-write these equations as

$$\text{(a)}\quad w''=-\frac{\partial U_1}{\partial w},\qquad U_1=-2w^3 \tag{7.12}$$

$$\text{(b)}\quad w''=-\frac{\partial U_2}{\partial w},\qquad U_2=-2w^3+\frac{w}{24} \tag{7.13}$$

and consider them as the equations of motion of a mass point on a line under an external force with potential energy $U_i(w)$ $(i=1,2)$. Then it becomes clear that the condition $w(t) \to_{t\to\infty} -1/12$ cannot be satisfied by a solution of Eq. (7.13); thus $\mu=-1$ does not lead to the correct asymptotics. Similarly, in the case $\mu=-6$ we need a solution of Eq. (7.12) such that $t^2 w(t) \to_{t\to\infty} 0$ and this is impossible.

Hence there are only three relevant values of μ:

$$\mu = 2/3; \qquad \mu = 3/2; \qquad \mu = -1/6$$

In the first two cases the solution of Eq. (7.12) must be chosen in such a way that $t^2 w(t) \to_{t\to\infty} 1$. General properties of the Weierstrass function $\mathscr{P}(t;\, 0,\, g_3)$ show that the only possible choice is $g_3=0$ and

$$w(t) = \mathscr{P}(t;\, 0,\, 0) = \frac{1}{t^2}, \qquad \mu = 2/3, \quad \mu = 3/2 \tag{7.14}$$

where, of course, we can change t to $t+t_0$ with an arbitrary $t_0>0$.

In the case $\mu=-1/6$, we need to satisfy the condition $t^2 w(t) \to_{t\to 0} 0$. This condition is satisfied by a wider class of solutions $w(t)=\mathscr{P}(t+t_0;\, 0,\, g_3)$ with arbitrary values of $t_0>0$ and g_3. The simplest solution of this type is

$$w(t) = \frac{1}{(t+t_0)^2}, \qquad t_0>0, \quad \mu=-1/6$$

The corresponding solution of Eq. (7.3) reads

$$y(p) = 1 - \frac{1}{(1+t_0 p)^2}, \qquad \mu = -1/6 \tag{7.15}$$

The previous two exact solutions (see (7.14)) provide two solutions of Eq. (7.3) in the following form:

$$y(p) = \frac{1}{(1+t_0 p^{-\beta})^2}; \quad \beta=1/2, \quad \mu=2/3; \quad \beta=1/3, \quad \mu=3/2 \tag{7.16}$$

The first solution (7.15) corresponds to the well-known BKW mode for the Boltzmann equation.[8–10] The other two cases of closed form solution (7.16) lead to the new similarity solutions found by Bobylev and Cercignani.[11, 12] Our investigation here shows that there is practically no hope to find any other closed form solution of Eq. (7.1).

Let us now consider the other two solutions (7.16) found above and invert the Laplace transform. The parameter t_0 reflects the fact that Eqs. (7.1) and (7.3) are invariant under scaling transformations; therefore it is sufficient to consider the case $t_0 = 1$. Then the solutions of Eq. (7.1) read as follows:

$$u(x) = \mathscr{L}^{-1}\left[\frac{1}{p}\frac{1}{(1+p^{-\beta})^2}\right] = \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)\, x^{n\beta}}{\Gamma(n\beta+1)},$$

$$\beta = 1/2\ (\mu = 2/3), \qquad \beta = 1/3\ (\mu = 3/2)$$

This can be seen by taking the Laplace transform, term by term, of the series for u_β. The process is easily justified for $|p| > 1$, but the result is valid for any p in the complex plane cut along the negative real semi-axis by analytic continuation. We can see now that these solutions belong to the general class of self-similar solutions discussed in detail in Section 6.

Methods developed in our previous paper[12] allow a much more convenient representation:

$$u_\beta(x) = 2\frac{\sin \beta\pi}{\beta\pi}\int_0^\infty \frac{ds(1+s\cos\beta\pi)}{(1+s^2+2s\cos\beta\pi)^2}\, e^{-xs^{-1/\beta}}, \qquad \beta = 1/2,\ 1/3$$

Then we remark that $x = |\mathbf{k}|^2/2$ and that

$$e^{-\Theta\,|\mathbf{k}|^2/2} = \mathscr{F}[M_\Theta] = \int_{R^d} d\mathbf{v}\, M_\Theta(|\mathbf{v}|)\exp(-i\mathbf{k}\cdot\mathbf{v}) \tag{7.17}$$

where $M_\Theta(|\mathbf{v}|)$ denotes the Maxwellian distribution

$$M_\Theta(|\mathbf{v}|) = (2\pi\Theta)^{-3/2}\, e^{-|\mathbf{v}|^2/(2\Theta)} \tag{7.18}$$

with "temperature" $\Theta > 0$. Hence the corresponding self-similar solutions of the Boltzmann equation (3.2) read:

$$f_\beta(|\mathbf{v}|, t) = e^{-\frac{3}{2}\mu_\beta t} F_\beta(|\mathbf{v}|\, e^{-\frac{1}{2}\mu_\beta t})$$

$$F_\beta(|\mathbf{v}|) = 2\frac{\sin\beta\pi}{\beta\pi}\int_0^\infty \frac{ds(1+s\cos\beta\pi)}{(1+s^2+2s\cos\beta\pi)^2} M_{\Theta(s)}(|\mathbf{v}|), \qquad \beta = 1/2,\ 1/3$$

where $\Theta(s) = s^{-1/\beta}$. These solutions are obviously integrable functions.

We note that

$$e^{-\frac{3}{2}\mu t} M_\theta(|\mathbf{v}|\, e^{-\frac{1}{2}\mu t}) = M_{\theta(t)}(|\mathbf{v}|), \qquad \theta(t) = \theta e^{\mu t};$$

$$M_{\theta_1}(|\mathbf{v}|)^*\, M_{\theta_2}(|\mathbf{v}|) = M_{\theta_1+\theta_2}(|\mathbf{v}|)$$

Hence, the extended class (see the end of Section 6) of invariant solutions reads

$$f_{\beta,\gamma}(|\mathbf{v}|, t) = 2\,\frac{\sin\beta\pi}{\beta\pi}\int_0^\infty \frac{ds(1+s\cos\beta\pi)}{(1+s^2+2s\cos\beta\pi)^2}\, M_{\theta(s,\,t;\,\beta,\,\gamma)}$$

where

$$\theta(s, t; \beta, \gamma) = s^{-1/\beta}e^{\mu_\beta t} + \gamma, \qquad \gamma > 0, \quad \beta = 1/2,\ 1/3$$

Finally we construct an example needed in the proof of Theorem 6.3. The so-called Mittag–Leffler function,[20]

$$\phi(x) = \sum_0^\infty \frac{(-1)^n}{\Gamma(1+n\alpha)}\, x^{n\alpha}, \qquad 0 < \alpha < 1 \tag{7.19}$$

has a relatively simple Laplace transform

$$\mathscr{L}[\phi] = \frac{p^{\alpha-1}}{1+p^\alpha}$$

and possesses the following integral representation:[12]

$$\phi(x) = \frac{\sin\alpha\pi}{\alpha\pi}\int_0^\infty \frac{ds\, e^{-xs^{-1/\alpha}}}{1+s^2+2s\cos\pi\alpha}, \qquad 0 < \alpha < 1 \tag{7.20}$$

Equation (7.20) obviously shows that $\phi(|\mathbf{k}|^2/2)$ is the Fourier transform of a positive function of $|\mathbf{v}|$. Hence the function (7.19) can be used as an example of characteristic function represented in the form (6.2) and this completes the proof of Theorem 6.3.

8. MORE ON ETERNAL SOLUTIONS OF THE BOLTZMANN EQUATION

We have constructed in Sections 6 and 7 some examples of non-negative eternal solutions of the Boltzmann equation. All the solutions have, however, an infinite second moment (energy). Can we construct any example of eternal solution with finite energy? What can be said about the properties of "usual" solutions f (with finite moments of all orders) when we extend them to negative values of t? This section is devoted to discussing these questions.

We consider again just isotropic solutions $f(|\mathbf{v}|, t)$ and assume (in the present section) that all moments

$$m_n(t) = \int_{\mathfrak{R}^3} d\mathbf{v}\, f(\mathbf{v}, t)\, |\mathbf{v}|^{2n}, \qquad n = 0, 1,... \tag{8.1}$$

are finite for $t \geqslant 0$. Furthermore we assume $m_0 = 1$, $m_1 = 3$ (conservation laws); the corresponding Maxwellian reads:

$$M(|\mathbf{v}|) = (2\pi)^{-3/2}\, e^{-|\mathbf{v}|^2/2} \tag{8.2}$$

We use again the Fourier transform (4.1), and consider the function $\phi(x, t)$ (5.1) satisfying Eq. (5.2). We slightly change the notation (5.6) and represent $\phi(x, t)$ as

$$\phi(x, t) = \sum_{n=0}^{\infty} \phi_n \frac{(-1)^n x^n}{n!}, \qquad \phi_0 = 1, \quad \phi_1 = 1, \quad \phi_n = \frac{m_n(t)}{(2n+1)!!} \tag{8.3}$$

The inequalities $\phi_n(t) > 0$, $n = 2, 3,...$ give a necessary condition for $f(|\mathbf{v}|, t)$ to be positive. It is convenient to introduce a new unknown function $\psi(x, t)$ by setting

$$\phi(x, t) = e^{-x}\psi(x, t); \qquad \psi(x, t) = \sum_{n=0}^{\infty} \psi_n \frac{(-1)^n x^n}{n!}, \quad \psi_0 = 1, \ \psi_1 = 0$$

$$\phi_n(t) = \sum_{k=0}^{n} \binom{n}{k} \psi_k(t), \qquad n = 0, 1,... \tag{8.4}$$

Then $\psi(x, t)$ satisfies the same equation (5.2)

$$\psi_t = \int_0^1 ds\, G(s)[\psi(sx)\, \psi((1-s)\, x) - \psi(0)\, \phi(x)] \tag{8.5}$$

and the initial condition

$$\psi_{|t=0} = \psi_0(x) = e^x \phi_{|t=0} \tag{8.6}$$

Equations for $\psi_n(t)$ follow from (8.4), (8.5):

$$\psi_0 = 1, \quad \psi_1 = 0; \qquad \frac{d\psi_i}{dt} - \lambda_i \psi_i = 0, \quad i = 2, 3$$

$$\frac{d\psi_n}{dt} - \lambda(n)\, \psi_n = \sum_{j=2}^{n-2} B_1(j, n-j)\, \psi_j \psi_{n-j}, \qquad n = 4, 5,... \tag{8.7}$$

in the notation (5.14). We also notice that the values

$$\phi^*(x) = e^{-x}, \quad \psi^*(x) = 1; \qquad \phi_n^* = 1. \qquad \psi_n^* = 0, \quad n = 1, 2, \ldots$$

correspond to the Maxwellian (8.2). Now we can prove the first result of this section.

Theorem 8.1. Let $f(|\mathbf{v}|, t)$ be an eternal solution of the Boltzmann equation such that $0 < m_n(t) < \infty$ ($n = 0, 1, \ldots$; see (8.1)) for all $-\infty < t < \infty$. Then $f(|\mathbf{v}|, t) = f(|\mathbf{v}|, 0)$ is a Maxwellian distribution function.

Proof. We assume without loss of generality that $m_0 = 1$, $m_1 = 3$. Then the only possible Maxwellian is given by Eq. (8.2). If $f(|\mathbf{v}|, 0) \neq M(|\mathbf{v}|)$, them $\psi_0(x) \neq 1$ in Eq. (8.6). We consider Eqs. (8.7) with the most general initial condition ($p \geqslant 2$):

$$\begin{gathered} \psi_{i|t=0} = 0, \quad 1 \leqslant i \leqslant p-1; \qquad \psi_{p|t=0} = \psi_p^{(0)} \neq 0; \\ \psi_{n|t=0} = \psi_n^{(0)}, \qquad n = p+1, \ldots \end{gathered} \tag{8.8}$$

It is easy to find $\psi_n(t)$ with $2 \leqslant n \leqslant 2p$, since, by recursion

$$\psi_i(t) = 0, \qquad 1 \leqslant i \leqslant p-1$$

and

$$\frac{d\psi_n}{dt} - \lambda(n)\,\psi_n = 0, \quad n = p\ldots, 2p-1; \qquad \frac{d\psi_{2p}}{dt} - \lambda(2p)\,\psi_{2p} = B_1(p, p)\,\psi_p^2$$

Therefore we obtain

$$\begin{gathered} \psi_i(t) = \psi_i^{(0)} e^{\lambda(i)\,t}, \quad i = p, \ldots, 2p-1; \qquad \psi_{p|t=0} = \psi_p^{(0)} \neq 0 \\ \psi_{2p}(t) = [\psi_{2p}^{(0)} + A]\, e^{\lambda(2p)\,t} - A e^{2\lambda(p)}, \qquad A = \frac{B_1(p, p)}{\lambda(2p) - 2\lambda(p)}\, [\psi_p^{(0)}]^2 > 0 \end{gathered} \tag{8.9}$$

The inequality $A > 0$ follows from the general properties of eigenvalues (5.14):[19]

$$\lambda(n) < 0, \qquad \lambda(n+1) < \lambda(n), \qquad \lambda(n) + \lambda(m) < \lambda(n+m), \qquad n, m = 2, \ldots$$

Let us consider now the coefficient $\phi_{2p}(t)$ of the series (8.3). By using Eqs. (8.4), (8.9), we obtain:

$$\phi_{2p} = \sum_{k=p}^{2p-1} \binom{n}{k} \psi_k^{(0)} e^{\lambda(k)\, t} + [\psi_{2p}^{(0)} + A]\, e^{\lambda(2p)\, t} - A e^{2\lambda(p)\, t}, \qquad A > 0$$

Noting that

$$\lambda(p) < 0, \qquad |2\lambda(p)| > \max_{p \leqslant k \leqslant 2p} |\lambda(k)|$$

we conclude that there exists a $T > 0$ such that

$$\phi_{2p} = \frac{m_{2p}(t)}{(4p+1)!!} < 0, \qquad \text{if} \quad t < -T$$

This contradicts the assumption of the theorem and hence $f(|\mathbf{v}|, 0) = M(|\mathbf{v}|)$. Hence $f(|\mathbf{v}|, t) = M(|\mathbf{v}|)$, since $M(|\mathbf{v}|)$ is a steady solution of the Boltzmann equation (3.2). The proof is completed.

Remark. The moment equations (8.7) always hold independently of the convergence or divergence of the series (8.3). The only condition for the nth equation to hold is the boundedness of $m_k(t)$ for $0 \leqslant k \leqslant n$ at some time instant, say at $t = 0$.

Thus any positive initial condition $f(|\mathbf{v}|, 0) \neq M(|\mathbf{v}|)$ possessing all the moments leads to a solution $f(|\mathbf{v}|, t)$ which cannot be positive for all $t < 0$. The solution $f(|\mathbf{v}|, t)$ can, of course, blow up at some $t_0 < 0$; this seems to be the most typical behavior.

On the other hand, eternal (nonpositive) solutions with all moments finite do exist, as was shown in ref. 12. We discuss some asymptotic properties of such solutions for $t \to -\infty$.

In order to study Eqs. (8.5)–(8.7) for $t < 0$, we temporarily denote $\tilde{t} = -t$ and then omit tildas. We obtain:

$$\begin{aligned} &\frac{d\psi_i}{dt} + \lambda_i \psi_i = 0, \qquad i = 2, 3 \\ &\frac{d\psi_n}{dt} + \lambda(n)\, \psi_n = -\sum_{j=2}^{n-2} B_1(j, n-j)\, \psi_j \psi_{n-j}, \qquad n = 4, 5,..., \quad t > 0 \end{aligned} \tag{8.10}$$

and again consider the most general initial conditions (8.8), following the same idea as in Section 5. By setting

$$\psi_n(t) = u_n e^{n\mu_p t}, \qquad \mu_p = -\frac{\lambda_p}{p} > 0 \tag{8.11}$$

with the same values $p \geqslant 2$ as in Eqs. (8.8), we obtain

$$\begin{aligned} &\frac{du_k}{dt} + \gamma_p(k)\, u_k = 0, \quad k = 2,..., 2p-1; \qquad \gamma_p(k) = k\mu_p + \lambda(k) \\ &\frac{du_n}{dt} + \gamma_p(n)\, u_n = -\sum_{j=p}^{n-p} B_1(j, (n-j))\, u_j u_{n-j}, \qquad n = 2p,..., \quad t > 0 \end{aligned} \tag{8.12}$$

since $u_1(t) = \cdots = u_{p-1}(t) = 0$ because of the initial condition (8.8). It can be shown (by considerations similar to those in ref. 19) that

$$\gamma_p(p) = 0, \qquad \gamma_p(n) > 0 \qquad \text{if} \quad n \geqslant p+1, \quad p \geqslant 3$$

Therefore we construct the solution of Eqs. (8.12) in the same way as we did in Section 5. This leads to the following asymptotics for $t \to \infty$:

(1) If $p \geqslant 3$, then $u_n(t) \to_{t\to\infty} 0$ if $n \neq mp$, $m = 1, 2,...$. If $n = mp$ then $u_{mp}(t) \to_{t\to\infty} y_m^{(p)}$, where $y_1^{(p)} = \psi_p^{(0)} \neq 0$, and $y_m^{(p)}$ with $m \geqslant 2$ is defined by the following recurrence relations:

$$y_m^{(p)} = -\frac{1}{\gamma_p(pm)} \sum_{j=1}^{m-1} B_1(jp, (m-j)\, p)\, y_j^{(p)} y_{m-j}^{(p)} \tag{8.13}$$

(2) If $p = 2$, then $u_n(t) \to_{t\to\infty} y_n$, $n = 2, 3,...$, where

$$y_2 = \psi_2 \neq 0, \quad y_3 = \psi_3 \neq 0; \quad y_n = -\frac{1}{\gamma_2(n)} \sum_{j=2}^{n-2} B_1(j, n-j)\, y_j\, y_{n-j}, \qquad n = 4,... \tag{8.14}$$

The special case $p = 2$ arises because of the well-known degeneracy[19] $\mu_2 = \mu_3$. We discuss below these results in the notation (8.5) for positive and negative t.

Remark. The above asymptotics becomes almost obvious if we remark that Eqs. (8.13), (8.14) describe a unique nontrivial steady solution of Eqs. (8.12) related to the initial condition (8.8).

We discuss below the results in the "normal" notation (8.5) for positive and negative values of t.

Theorem 8.2. Let $f(\mathbf{v}, t)$ satisfy the Boltzmann equation (3.2) for all $t < 0$. We assume that $(1+|\mathbf{v}|^{2N})\, f(\mathbf{v}, t) \in L^1(\Re^3)$, $N \geqslant 2$, and set

$$\phi_n(t) = \frac{m_n(t)}{(2n+1)!!}, \qquad n = 0, 1,..., N, \quad t \leqslant 0$$

in the notation (8.3).Then the initial conditions satisfying

$$\phi_0(0) = \phi_1(0) = \cdots = \phi_{p-1}(0) = 1, \qquad \phi_p(0) \neq 1$$

for a given $2 \leqslant p \leqslant N$ lead to the following asymptotics

$$\phi_0 = \phi_1 = 1, \qquad \lim_{t \to -\infty} e^{n\mu_p t} \phi_n(t) = y_n, \qquad n = 2,..., N$$

where

$$\mu_p = \frac{1}{p} \int_0^1 ds\, G(s)[1 - s^p - (1-s)^p] > 0$$

and y_n is given by the recurrence formulas (8.13) for $p \geqslant 3$ and (8.14) for $p = 2$.

Proof. First we note that (see (8.4))

$$\phi_n(t) = \sum_{k=0}^{n} \binom{n}{k} \psi_k(t), \qquad n = 0, 1,..., N$$

and that Eq, (8.7) for $\psi_n(t)$ hold even if the higher moments (with $n > N$) do not exist finite. Eqs. (8.11) lead to

$$\phi_n(t) = \sum_{k=0}^{n} \binom{n}{k} u_k(t)\, e^{k\mu_p |t|}, \qquad n = 0, 1,..., N, \quad t < 0$$

therefore

$$\lim_{t \to -\infty} e^{n\mu_p t} \phi_n(t) = \lim_{t \to -\infty} u_n(|t|) + \sum_{k=0}^{n-1} \binom{n}{k} \lim_{t \to -\infty} u_k(t)\, e^{-(n-k)\,\mu_p\,|t|}, \quad n = 2,..., N$$

It has been explained already how to prove the asymptotic formulas for $u_n(|t|)$, and we can conclude provided we admit that $\gamma_p(n) > 0$ if $n \geqslant p+1$, $p \geqslant 3$. For brevity, the elementary proof of this inequality, based on the convexity of $\lambda(p)$ (5.14) and on the equality $\mu_2 = \mu_3$ is left to the reader. This completes the proof of the theorem.

Theorem 8.2 applied to the case $N=\infty$ (all moments exist finite) shows that formally

$$f(\mathbf{v}, t) \cong_{t \to -\infty} e^{\frac{3}{2}\mu_p t} F(|\mathbf{v}|\, e^{\frac{1}{2}\mu_p t}) \tag{8.15}$$

where

$$\frac{1}{(2n+1)!!} \int_{\mathfrak{R}^3} d\mathbf{v}\, F(|\mathbf{v}|)\, |\mathbf{v}|^{2n} = y_n, \qquad n=2... \tag{8.16}$$

and the convergence in Eq. (8.15) is understood in the sense of convergence of all moments. The asymptotic equality (8.15) makes sense provided a function $F(|\mathbf{v}|)$ with the assigned moments y_n exists. The existence of such an $F(|\mathbf{v}|)$, as we shall see below, can be easily proved just for one special case (BKW mode). In all other cases, however, it seems quite probable that an $F(|\mathbf{v}|)$ satisfying (8.16) does not exist. This, in turn, makes it doubtful that the corresponding solutions $f(|\mathbf{v}|, t)$ exist for all $t<0$, i.e., do not blow up at some $t=-T<0$.

In order to examine the existence of $F(|\mathbf{v}|)$, we note that formally (since we do not know if $F(|\mathbf{v}|)$ exists) we have

$$\int_{\mathfrak{R}^3} d\mathbf{v}\, F(|\mathbf{v}|)\, e^{-i\mathbf{k}\cdot\mathbf{v}} = y(|\mathbf{k}|^2/2) \tag{8.17}$$

where

$$y(x) = \sum_{n=0}^{\infty} y_n \frac{(-1)^n x^n}{n!}, \qquad y=1, \quad y_1=1 \tag{8.18}$$

and y_n, $n \geqslant 2$, are given by Eqs. (8.13) ($p \geqslant 3$) or Eqs. (8.14) ($p=2$). It is convenient to use the notation (8.18) for $p=2$ and the notation

$$y^{(p)}(x) = 1 + \sum_{n=1}^{\infty} y_n^{(p)} \frac{(-x)^{np}}{(np)!} \tag{8.19}$$

for $p \geqslant 3$ (in agreement with Eqs. (8.13), (8.14)).

We note that

$$\psi(x, t) = y(xe^{-\mu_2 t}), \quad \psi^{(p)}(x, t) = y^{(p)}(xe^{-\mu_p t}), \qquad p \geqslant 3 \tag{8.20}$$

are self-similar solutions of Eq. (8.5) constructed and studied long ago.[4, 6] In particular, it is known that the functions (8.18), (8.19), extended to complex values of x, are entire analytic function of the exponential type.

Roughly speaking, a typical asymptotic behavior of such functions is given by the asymptotic equality

$$\log |y(re^{i\theta})| \cong_{r\to\infty} \alpha(\theta)\, r, \qquad 0 \leqslant \theta < 2\pi$$

On the other hand, the case $\alpha(0) \geqslant 0$ contradicts the assumption (8.17). Hence we need to study the behavior of the functions (8.18), (8.19) for large positive x. There is an essential difference between the cases $p = 2$ and $p \geqslant 3$. Equations (8.12) for $p \geqslant 3$ contain just one free parameter $y_1^{(p)}$. Furthermore

$$y^{(p)}(x) = 1 - \sum_{n=1}^{\infty} \frac{a_n^{(p)}}{(np)!} [(-1)^{p+1} y_1 x^p]^n \tag{8.21}$$

where $a_n^{(p)} > 0$ are given by the following equalities:

$$a_1 = 1, \qquad a_n^{(p)} = \frac{1}{\gamma_p(pn)} \sum_{j=1}^{n-1} B_1(j, n-j)\, a_j^{(p)} a_{(n-j)\,p}^{(p)}, \qquad n = 2,...$$

for $p \geqslant 3$. Hence $y^{(p)}(x)$ can be bounded just in the case when $(-1)^{p+1}\, y_1 < 0$ provided the function (8.21) with $y_1 = (-1)^p$ is bounded. Thus in the case $p \geqslant 3$ everything depends on the boundedness of some function having no free parameters.

The case $p = 2$ is special since in Eq. (8.13) we have two parameters y_2 and y_3. One of them merely corresponds to the change of variables $x \to \alpha x$, $\alpha = \text{const.}$, and in this there is no difference with the case $p \geqslant 3$. For $p = 2$, however, we have a further parameter, which can be used to control the asymptotic behavior for large x. And we already know (BKW mode from ref. 11) the combination of y_1 and y_2 leading to the bounded solution

$$y(x) = e^{-ax}(1 + ax), \qquad a > 0, \qquad y_2 = -a^2/2, \qquad y_3 = -a^2/3 \tag{8.22}$$

The corresponding self-similar solution $\psi(x, t) = y(xe^{-\mu_2 t})$ is universal: it satisfies Eq. (8.5) for any kernel $G(s)$. In the case $G(s) = 1$ considered in detail in Section 7, it can be proved on the basis of ODE (7.3) that the function (8.22) is the unique bounded function in the whole class given by Eqs. (8.12), (8.13). It seems quite probable that this is also true for the general kernel $G(s)$ though we did not prove this.

Hence, at least in the case $G(s) = 1$, the asymptotic equality (8.15) makes sense only if $p = 2$ and $y(x)$ is given by Eq. (8.22). Then the set of eternal solutions $f(|\mathbf{v}|, t)$ of this kind (to be eternal, they must, of course, be defined for $t > 0$ as well) is rather small, since the initial conditions must satisfy the restrictions (8.22) for $\psi_i^{(0)} = y_i$, $i = 2, 3$. On the other hand, this shows a specific role of the BKW self-similar solution (8.20), (8.22) as the only possible asymptotic state for the eternal solutions with finite moments of all orders. This conclusion can be considered as an answer to the so-called Krook–Wu conjecture[14] which was extensively discussed in the 1970s–1980s: does this simple solution play any special (asymptotic) role? The answer we give is "yes," but in the non-physical domain of large negative times and partly negative solutions.

Finally we note that the function $\phi_a e^{-\Theta x}$, $\Theta > 0$, where $\phi_a(x, t)$ is given by Eqs. (8.20), (8.22), is also an exact solution of Eq. (5.2). Therefore we obtain the following one-parameter family of eternal solutions of the Boltzmann equation (3.2)

$$f_\Theta(|\mathbf{v}|, t) = \left[1 + \frac{e^{-\mu t}}{\Theta(t)} \left(3 - \frac{|\mathbf{v}|^2)}{\Theta(t)} \right) \right] M_{\Theta(t)}(|\mathbf{v}|)$$

where $M_\Theta(|\mathbf{v}|)$ is given by Eq. (7.18),

$$\Theta(t) = \Theta + e^{-\mu t}, \qquad \mu = \int_0^1 ds\, G(s)\, s(1-s), \qquad \Theta \geqslant 0, \quad t \in (-\infty, \infty)$$

We do not know whether any other nontrivial eternal solutions having finite moments exist. The above (heuristic) arguments show that such solutions, if they exist, must have the same asymptotic behavior (for $t \to -\infty$) as the exact solutions $f_\Theta(|\mathbf{v}|, t)$.

9. CONCLUDING REMARKS

As has been noticed by many authors, starting with McKean,[23] there is a strong analogy between the central limit theorem and the trend to equilibrium for the spatially homogeneous Boltzmann equation. This analogy is best seen in the case of Maxwellian molecules, because of the similarity between the action of the gain term with its properties similar to those of a convolution, which enable us to mimic some of the important inequalities and proofs of the central limit theorem, in the context of the Boltzmann equation.

We have investigated the question : what is the asymptotic behavior of the BE when the energy is infinite?

In probability theory the analogous problem is the case of infinite variance. Under this assumption, the right scaling is not, like in classical central limit theorem,

$$(X_1 + X_2 + X_3 + \cdots + X_n)/\sqrt{n}$$

but

$$(X_1 + X_2 + \cdots + X_n)/n^\beta$$

for some exponent β which depends on the tail behavior of the independent random variables X_n. Then the convergence is not towards a Gaussian, but towards one of the so-called stable laws. The simplest of these are due to Levy, and in the symmetric case their Fourier transforms look like $e^{-c\,|\mathbf{k}|^\alpha}$ for some α which is between 0 and 2.

When one considers the Boltzmann equation with infinite energy, it is quite possible that the solution does not converge for long times to any reasonable distribution. However, since the energy is infinite, there is another degree of freedom: one can rescale the velocity space in whatever way, preserving the mass and momentum of the solution. This operation of rescaling is time-dependent and is the analogue of the fact that when the variance is infinite, the scaling for sums of random variables is n-dependent (n is analogous to an exponential of the time variable in the Boltzmann equation). We have seen that this rescaled solution converges to something: the self-similar solutions, when rescaled, are like equilibrium distributions.

One can hope that these solutions may give qualitative information on the profile of some unsteady situations where the energy is so high, that the convergence to equilibrium only holds on a very slow scale.

We also remark that Toscani and Villani[24] considered the space homogeneous Boltzmann equation for Maxwell molecules (without cutoff) and proved a uniqueness result, which appears to be the first result of this kind available for long-range interactions. To this end they introduced metrics for probability measures on $\mathfrak{R}_d$ with a vanishing first moment and given second moment. The distance d_s between two distributions with Fourier transforms $\hat{f}(\mathbf{k})$ and $\hat{g}(\mathbf{k})$ was defined as

$$d_s = \sup_{k \in \mathfrak{R}_d} \left(|\hat{f} - \hat{g}|\;|\mathbf{k}|^{-s}\right)$$

They proved that the distance d_2 between two solutions of the homogeneous Boltzmann equation cannot increase with time; uniqueness follows as a corollary. The distance d_2 was also applied to the central limit theorem, which was proved with an explicit estimate of the n_0 such that the distance d_2 between the nth term and the limit is less than ϵ for $n \geqslant n_0$.

It seems reasonable that similar properties hold for our solutions if we use d_s (with a suitable s) rather than d_2.

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